Duality Theory

~~To every L.P.P there~~

Associated with every L.P.P. there exists a corresponding L.P.P. The original L.P.P, is known as the primal problem and the corresponding problem is known as the dual problem

Standard form of primal

A L.P.P is said to be in standard form if

1. All constraints involve the sign '$\leq$' in a problem of maximization or
2. All constraints involve the sign '$\geq$' in a problem of minimization.

Formulation of primal-dual problems

Let the standard primal problem in L.P.P be

maximize $z = \sum_{j=1}^{n} c_j x_j$

Subject to, $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$

Where $i = 1, 2, \ldots m$; $j = 1, 2, \ldots n$. Then the associated dual problem is given by —

minimize $w = \sum_{i=1}^{m} b_i v_i$

Subject to, $\sum_{i=1}^{m} a_{ij} v_i \geq c_j$

Where $v_1, v_2, \ldots v_m$ are dual variables

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and W is the dual objective function.

In detainls, if the primal problem be

$$\max Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$

then its dual problem will be

$$\text{minimize } W = b_1v_1 + b_2v_2 + \cdots + b_mv_m$$

subject to,

$$a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m \geq c_1$$
$$a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m \geq c_2$$
$$\cdots$$
$$a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m \geq c_n$$

where $v_i \geq 0$ for $(i = 1, 2, \ldots m)$

Note:- In vector notations, a primal and the corresponding dual problem may be written as follows ——

$$\left.\begin{array}{l} \text{maximize } Z = CX \\ \text{subject to } AX \le b \\ \text{where } X \geq 0 \end{array}\right\} \text{—} (1)$$

and its dual problem be

minimize $\omega = b^T v$

Subject to, $A^T v \geq c^T$

and $v \geq 0$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$c = [c_1, c_2, \cdots, c_n]$; $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$v = [v_1, v_2, \cdots, v_m]$

and $X = [x_1, x_2, \cdots, x_n]$

Remarks –

(1) The number of variables in the dual is the same as the number of constraints in the primal problem and vice-versa.

(2) If the primal (dual) problem is a maximization problem then its dual (primal) is a minimization problem.

(3) The variables in both the problems are non-negative.

(4) The columns of the co-efficient matrix are the activity vectors of the primal problem and on the other hand, the rows of the

co-efficient matrix are the activity vectors of the dual problem.

(5) To each primal constraint there will be a dual variable, i,e, if v_i ($i=1,2,\dots m$) be the i-th dual variable, then this will correspond to the i-th primal constraint.

(*) Rules for forming dual problem from a primal problem in L.P.P

The following steps are generally followed —

Step-1 :- Any primal problem of an L.P.P is first put in the standard primal form i,e, of the form

$$\text{maximize } z = cx$$
$$\text{subject to } Ax \leq b$$
$$\text{where } x \geq 0$$

Step-2 :- In this step the following instruction are followed —

(a) The maximization problem in the primal is transfered to a minimization problem in the dual.

(b) If the primal problem is with n variables and m constraints, then the corresponding dual problem is with m variables and n

constraints.

(c) n prices $c_1, c_2, \cdots c_n$ in the objective function of the primal problem become the requirements of the dual problem and m requirements $b_1, b_2, \cdots b_m$ of the primal problem become the prices of the dual objective function.

(d) The '$\le$' signs of the primal constraints becomes '$\geq$' signs of the dual constraints.

Illustrative Example :

(1) Write the dual of the following L.P.P

maximize $z = x_1 - x_2 + x_3$

subject to,

$$x_1 + x_2 + x_3 \le 10$$

$$2x_1 - x_3 \le 2$$

$$2x_1 - 2x_2 + 3x_3 \le 6$$

$$(x_1, x_2, x_3 \geq 0)$$

Solution :- The given primal problem is already in standard form i,e, of the form

maximize $z = cx$

subject to, $Ax \le b$; $x \geq 0$

If v_1, v_2, v_3 be the dual variables, then the corresponding dual problem will be given ~~for~~ from the following scheme —

$$\begin{array}{l} (1)\,x_1 + (-1)\,x_2 + (3)\,x_3 \\ \hline 1.x_1 + 1.x_2 + 1.x_3 \le 10 \rightarrow V_1 \\ 2x_1 + 0\,x_2 - 1.x_3 \le 2 \rightarrow V_2 \\ 2x_1 + (-2)\,x_2 + 3x_3 \le 6 \rightarrow V_3 \end{array}$$

the dual problem will be

minimize $w = 10V_1 + 2V_2 + 6V_3$

subject to,

$$V_1 + 2V_2 + 2V_3 \geq 1$$
$$V_1 + 0.V_2 - 2V_3 \geq -1$$
$$V_1 - V_2 + 3V_3 \geq 3.$$
$$(V_1, V_2, V_3 \geq 0)$$

(2) maximize $Z = 3x_1 + 2x_2 + x_3$

subject to,

$$x_1 - 3x_2 + 2x_3 \le 5$$
$$x_1 + 5x_2 + x_3 \geq 8$$
$$(x_1, x_2, x_3 \geq 0)$$

solution :- The given primal problem is not in standard form. First we have to convert it in standard form. The standard form will be

maximize $Z = 3x_1 + 2x_2 + x_3$

subject to,

$$x_1 - 3x_2 + 2x_3 \le 5$$
$$-x_1 - 5x_2 - x_3 \le -8$$
$$(x_1, x_2, x_3 \geq 0)$$

If v_1, v_2 be the dual variables, then the corresponding dual problem will be given from the following scheme —

$$
\begin{array}{l}
(3)x_1 + (2)x_2 + 1\cdot x_3 \\
\hline
1\cdot x_1 + (-3)x_2 + 2x_3 \leq 5 \rightarrow v_1 \\
(-1)x_1 + (-5)x_2 + (-1)x_3 \leq -8 \rightarrow v_2
\end{array}
$$

The dual problem will be —

minimize $w = 5v_1 - 8v_2$

subject to,

$$v_1 - v_2 \geq 3$$
$$-3v_1 - 5v_2 \geq 2$$
$$2v_1 - v_2 \geq 1 \quad (v_1, v_2 \geq 0)$$

(3) Find the dual of the following primal

minimize $z = 3x_1 - 2x_2$

subject to,

$$2x_1 + x_2 \leq 1$$
$$-x_1 + 3x_2 \geq 4$$
$$(x_1, x_2 \geq 0)$$

Solution:- The given primal problem is not in standard form. The standard form will be —

$$\min z = 3x_1 - 2x_2$$

subject to,

$$-2x_1 - x_2 \geq -1$$
$$-x_1 + 3x_2 \geq 4$$
$$(x_1, x_2 \geq 0)$$

If v_1, v_2 be the dual variables, then the corresponding dual problem will be given from the following scheme ←

$$\begin{array}{l|l} (3)x_1 + (-2)x_2 & \\ \hline (-2)x_1 + (-1)x_2 \geq -1 & v_1 \\ (-1)x_1 + (3)x_2 \geq 4 & v_2 \end{array}$$

Now required dual problem will be

max $w = -v_1 + 4v_2$

subject to, $-2v_1 - v_2 \leq 3$

$-v_1 + 3v_2 \leq -2$

$(v_1, v_2 \geq 0)$

(4) min $z = 3x_1 + x_2$

subjet to, $2x_1 + x_2 \geq 14$

$x_1 - x_2 \geq 4$ $(x_1, x_2 \geq 0)$

Solution :- Let v_1, v_2 be the dual variables. clearly the given primal problem is in standard form. so, its dual problem will be,

max $w = 14v_1 + 4v_2$

subject to, $2v_1 + v_2 \leq 3$

$v_1 - v_2 \leq 1$

$(v_1, v_2 \geq 0)$.

H.W

obtain the dual of the given L.P problems —

(1) maximize $Z = 2x_1 + 3x_2 - 4x_3$

subject to, $5x_1 - 2x_2 + x_3 \leq 4$

$x_1 + x_2 - 4x_3 \leq 7$

$(x_1, x_2, x_3 \geq 0)$

(2) min $Z = x_1 - 3x_2 + 2x_3$

subjet to, $3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$(x_1, x_2, x_3 \geq 0)$

(3) max $Z =$ ~~$2x_1 - 6x_2$~~ $2x_1 + 5x_2 + 6x_3$

subject to,

$5x_1 + 6x_2 - x_3 \leq 3$

$-2x_1 + x_2 + 4x_3 \leq 4$

$x_1 - 5x_2 + 3x_3 \leq 1$

$3x_1 + 3x_2 - 7x_3 \geq -6$

$(x_1, x_2, x_3 \geq 0)$

(4) min $z = 3x_1 - 2x_2$

subjet to, $2x_1 + x_2 \leq 1$

$-x_1 + 3x_2 \geq 4$ $(x_1, x_2 \geq 0)$.

Theorem

Dual of the dual is primal itself

proof:- Let the primal problem be —

maximize $Z = cX$
Subjet to, $AX \leq b \quad (X \geq 0)$ } — (1)

The dual of which is
minimize $Z^* = b^T w$
subject to $A^T w \geq c^T \quad (w \geq 0)$. } — (2)

The problem (2) is equivalent to the problem
max $(-b^T w)$
subjet to $-A^T w \leq -c^T \quad (w \geq 0)$ } — (3)
where $\min(b^T w) = -\max(-b^T w)$.

NOW the dual problem (3) looks like a primal problem (1) and hence considering it as a primal, the dual of it can be written as —

minimize $(-cX)$
subject to $-(A^T)^T X \geq (-b^T)^T \quad (X \geq 0)$
$\Rightarrow -AX \geq -b$
$\Rightarrow AX \leq b. \quad (X \geq 0)$

$\therefore$ max (cX)
subjet to, $AX \leq b \quad (X \geq 0)$

$\boxed{\text{Since } \min(-cX) = \max(cX)}$

which is ~~same as~~ of the form (1).
$\therefore$ Dual of dual is primal.

using the theory "dual of the dual is the primal" verify the following problem. (11)

1) maximize $Z = 2x_1 + x_2 - x_3$

subject to $4x_1 - x_2 + x_3 \leq 4$

$x_1 + 3x_2 + 4x_3 \leq 8$ $(x_1, x_2, x_3 \geq 0)$

solution:- This maximizing problem can be written in the manner

$$\max z_x = (2, 1, -1)[x_1, x_2, x_3]$$

subject to

$$\begin{bmatrix} 4 & -1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 8 \end{bmatrix} \quad (x_1, x_2, x_3 \geq 0)$$

The dual of it is,

$$\left.\begin{aligned} &\text{minimize } z_w = 4w_1 + 8w_2 \\ &\text{subjet to, } 4w_1 + w_2 \geq 2 \\ &\qquad -w_1 + 3w_2 \geq 1 \\ &\qquad w_1 + 4w_2 \geq -1 \end{aligned}\right\} \leftarrow (1)$$

$(w_1, w_2 \geq 0)$

(w_1, w_2 being dual variables)

Now considering the above (1) minimization problem as primal, the dual of it is

$$\max z_v = 2v_1 + v_2 - v_3$$

subject to, $4v_1 - v_2 + v_3 \leq 4$

$v_1 + 3v_2 + 4v_3 \leq 8$

$(v_1, v_2, v_3 \geq 0)$

Which is nothing but the primal problem. Hence proved

(2) min $Z = 3x_1 + 4x_3$ (12)

subject to, $4x_1 + 2x_2 - x_3 \geq 12$

$x_1 + 5x_2 + x_3 \geq 18$

$x_1 - x_2 + 7x_3 \geq 2 \quad (x_1, x_2, x_3 \geq 0)$

If The give primal problem is in standard form. Let w_1, w_2, w_3 be dual variables then dual be

max $Z_w = 12w_1 + 18w_2 + 2w_3$

subject to
$$\left.\begin{array}{l} 4w_1 + w_2 + w_3 \leq 3 \\ 2w_1 + 5w_2 - w_3 \leq 0 \\ -w_1 + w_2 + 7w_3 \leq 4 \end{array}\right\} - (1) \quad (w_1, w_2, w_3 \geq$$

Now considering (2) as primal problem. Dual of it is

min $Z_v = 3V_1 + 4V_3$

subject to, $4V_1 + 2V_2 - V_3 \geq 12$

$V_1 + 5V_2 + V_3 \geq 18$

$V_1 - V_2 + 7V_3 \geq 2$

$(V_1, V_2, V_3 \geq 0)$

Which is the given problem.

So, Dual of dual of a primal is primal itself.

A.W verify "Dual of the dual is the primal itself" for given four problems.

Importance of the Duality Theory

When the number of constraints are greater than the number of variables, duality theory is very helpful in solving the problem by simplex method. For example, let the number of constraints be five and the number of variables be two in the primal problem. If we try to solve the primal problem by simplex method, then the basis will be a (5×5) square matrix. and it requires enough time to compute in each table. But if we convert the problem into its dual, we get only two constraints instead of five and the dual problem can be solved easily. Now solving the dual problem we get the optimal value of the objective function of the primal as well as the primal optimal variables. Hence by using duality theory we can solve the problems (sometimes) easily and more quickly.

PS Any quaries please contact with me.

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(Any time)